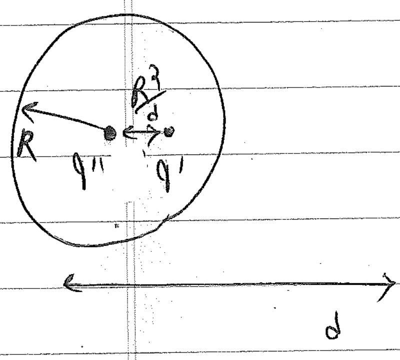
PHYC 511
Spring 2018

Problem Session 2 01/26/2018

(1) Problem 2.4, Jackson.

(2) Problem 2.8 Jackson.

(2) (a) The effect of the sphere is represented by two image charges as shown below:



$$q' = -\frac{a}{d} q \quad , \quad q'' = q + \frac{a}{d} q$$

The second charge q'', situated at the center, ensures that the surface remains equipotential while the total enclosed charge is q.

The net force on charge q is:

$$F = \frac{qq'}{4\pi\varepsilon_0 \left(d - \frac{R^2}{d}\right)^2} + \frac{qq''}{4\pi\varepsilon_0 d^2} = 0 \Rightarrow \boxed{\frac{d^5}{R^5} - 2\frac{d^3}{R^3} - 2\frac{d^2}{R^2} + \frac{d}{R} + 1 = 0}$$

to find the equilibrium point

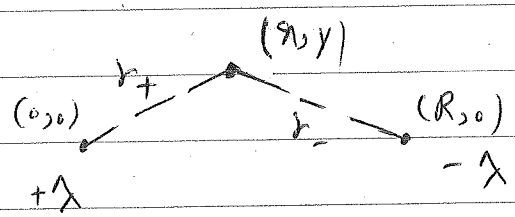
The equation can be solved numerically. We can check that $\boxed{\frac{d}{R} = 1.6187}$ is the solution. For larger values of d the force is repulsive as the charge q on the sphere has a larger effect, while for smaller value of d the force is attractive.

(b) Writing $d = R+a$, the attractive force close to the sphere is:

$$|F| \simeq \frac{q^2}{4\pi\varepsilon_0 \left(a+R-\frac{R^2}{a+R}\right)^2} = \frac{q^2(a+R)^2}{4\pi\varepsilon_0\left((R+a)^2-R^2\right)^2} \simeq \boxed{\frac{q^2R^2}{16\pi\varepsilon_0 a^2}}$$

(c) The answer in part (a) changes to $\boxed{\frac{d}{R} = 1.882}$. Part (b) remains unchanged though.

(2) (a) We have:



$$V(x,y) = \frac{-\lambda}{2\pi\varepsilon_0}\ln r_+ + \frac{\lambda}{2\pi\varepsilon_0}\ln r_- + C$$

Here C is a constant, which we can take to be zero. Then:

$$V(x,y) = \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_+}{r_-}\right)$$

Where:

$$r_+ = \sqrt{x^2+y^2}, \quad r_- = \sqrt{(x-R)^2+y^2}$$

For $V = V_0$, the equipotential surfaces are given by:

$$\frac{r_+}{r_-} = e^{-\frac{2\pi\epsilon_0 V_0}{\lambda}} \equiv \eta$$

For $V_0 > 0$, we have $\eta < 1$. This gives:

$$x^2 + y^2 = \eta^2 (x - R)^2 + \eta^2 y^2 \Rightarrow x^2(\eta^2 - 1) + y^2(\eta^2 - 1) - 2\eta^2 R x + \eta^2 R^2 = 0$$

$$\Rightarrow \left(x - \frac{\eta^2 R}{\eta^2 - 1}\right)^2 + y^2 = \left(\frac{\eta^4}{(\eta^2-1)^2} - \frac{\eta^2}{\eta^2 - 1}\right) R^2$$

This is the equation for circule centered at $\boxed{\left(\frac{\eta^2}{\eta^2 - 1} R, 0\right)}$

and Radius $\boxed{\tilde{R} = \left[\left(\frac{\eta^4}{(\eta^2-1)^2} - \frac{\eta^2}{\eta^2-1}\right) R^2\right]^{\frac{1}{2}}}$.

For $V_0 = -|V_0| < 0$, we find a circle with the same radius centered near the negative line charge. In this case $\eta > 1$.

(b) We take cylinder 1 to correspond to $\eta_1 < 1$ and cylinder 2 to correspond to $\eta_2 > 1$ (i.e., $V_1 > 0$ and $V_2 = -|V_2| < 0$). Then:

$$a = \frac{R \eta_1}{1 - \eta_1^2} = \frac{R}{2} \operatorname{cosech}\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right)$$

$$b = \frac{R \eta_2^{-1}}{1 - \eta_2^{-2}} = \frac{R}{2} \operatorname{cosech}\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right)$$

And:

(5)

$$d = \frac{R\eta_1^2}{1-\eta_1^2} + \frac{R\eta_2^2}{\eta_2^2-1} = \frac{R}{2}\left[\coth\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right) + \coth\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right)\right]$$

Now:

$$d^2 = \frac{R^2}{4}\left[1 + \operatorname{cosech}^2\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right) + 1 + \operatorname{cosech}^2\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right) + 2\coth\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right)\coth\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right)\right]$$

$$= a^2 + b^2 + \frac{R^2}{2}\,\frac{\left[\cosh\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right)\cosh\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right) + \sinh\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right)\sinh\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right)\right]}{\sinh\left(\frac{2\pi\epsilon_0 V_1}{\lambda}\right)\sinh\left(\frac{2\pi\epsilon_0 |V_2|}{\lambda}\right)}$$

$$= a^2 + b^2 + 2ab\cosh\left(\frac{2\pi\epsilon_0 (V_1 + |V_2|)}{\lambda}\right)$$

Note that $V_1 + |V_2| = V_1 - V_2 = \Delta V$. Hence:

$$\frac{2\pi\epsilon_0}{\lambda}\Delta V = \cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right) \Rightarrow \boxed{C = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)}}$$

(c) For $d \gg a, b$, we have $d^2 - a^2 - b^2 \gg 2ab$. This implies that:

$$y \equiv \frac{d^2-a^2-b^2}{2ab} \gg 1 \Rightarrow \cosh^{-1} y \simeq \ln(2y)$$

Therefore:

$$C \xrightarrow[d \gg a,b]{} \frac{2\pi\epsilon_0}{\ln\left(\frac{d^2}{ab}\right)} = \pi\epsilon_0\left(\ln\frac{d}{\sqrt{ab}}\right)^{-1}$$ (as in problem 1.7)

(d) When one cylinder is inside the other, both of η_1 and η_2 are >1 or <1. Assuming $V_1, V_2 > 0$, equivalently $\eta_1, \eta_2 < 1$, we have:

$$a = \frac{R}{2} \operatorname{Cosech} \frac{2\pi\varepsilon_0 V_1}{\lambda}, \quad b = \frac{R}{2} \operatorname{Cosh}\left(\frac{2\pi\varepsilon_0 V_2}{\lambda}\right)$$

$$d = \frac{R}{2}\left[\operatorname{Coth}\left(\frac{2\pi\varepsilon_0 V_1}{\lambda}\right) - \operatorname{Coth}\left(\frac{2\pi\varepsilon_0 V_2}{\lambda}\right)\right]$$

Similarly to part (b), we can show that:

$$\boxed{C = \frac{2\pi\varepsilon_0}{\operatorname{Cosh}^{-1}\left(\frac{a^2 + b^2 - d^2}{2ab}\right)}}$$